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THE MOD 2 COHOMOLOGY OF THE EXCEPTIONAL GROUPS*

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In this note we prove the mod 2 cohomology of an exceptional Lie group can be computed from the rational cohomology. In fact the rational cohomology determines the mod 2 cohomology as an algebra over the Steenrod algebra.

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The purpose of this note is to prove the following theorem.

Theorem. *Let X be a simply connected finite H-space with $H_*(X;\mathbb{Z}_2)$ an associative ring. Then if $H^*(X;\mathbb{Q})$ is isomorphic as algebras to the rational cohomology of an exceptional Lie group, then $H^*(X;\mathbb{Z}_2)$ is isomorphic as algebras over the Steenrod algebra to the* mod 2 *cohomology of an exeptional Lie group.*

This theorem shows that the rational cohomology of an exceptional Lie group determines the mod 2 cohomology. Several authors have previously used differentiable properties to compute the mod 2 cohomology [1, 2, 5]; many of the proofs are quite complex and depend on embeddings of exceptional groups in other groups, the action of the Weyl group, and fiberings over classical groups. The proof given here is strictly homological, and furthermore, some information about the coalgebra structure can also be deduced. (See Theorems 6 and 7).

Throughout the paper we will assume that all H-spaces mentioned will be simply connected, finite, and have associative mod 2 homology rings. The symbol X will denote such an H-space and we will use the following notation for these graded modules

$$H^* = H^*(X;\mathbb{Z}_2),\qquad Q^* = QH(X;\mathbb{Z}_2),\qquad P^* = PH^*(X;\mathbb{Z}_2).$$

We list here some theorems which will be useful for later computations.

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Theorem 1 [7]. *Let $R=\{x\in H^* \mid \bar{\Delta}x\in \xi H^*\otimes H^*\}$. Then R is a coalgebra over the Steenrod algebra and there is an exact sequence*

$$0\to \xi H^*\to R\to Q^*\to 0.$$

Furthermore, if I is the ideal in the symmetric algebra $S(R)$ generated by elements $\xi x - x\otimes x$, then H^* is isomorphic as Hopf algebras over the Steenrod algebra to $S(R)/I$.

From Theorem 1, we can conclude that the action of the Steenrod algebra on R determines the action on H^*.

Definition 1. Expand a positive integer n dyadically and assume 2^r is the first power of two missing in the dyadic expansion. So n has the form

$$n=1+2+\cdots+2^{r-1}+2^{r+1}k=2^r+2^{r+1}k-1$$

for $k\geq 0$. If $k>0$ we say n has *dyadic degree r.*

In the following theorem we show that all elements of dyadic degree r in R lie in the image of the Steenrod algebra.

Theorem 2 [9]. (a) *For $r>0$, $k>0$,*

$$R^{2^r+2^{r+1}k-1}=\mathrm{Sq}^{2^r k}R^{2^r+2^r k-1},$$

For $r=0$, $k>0$,

$$R^{2k}\subseteq \xi H^*.$$

(b) *For $r\geq 0$, $k>0$,*

$$\mathrm{Sq}^{2^r}R^{2^r+2^{r+1}k-1}=0.$$

(c) *Let $\sigma^*: H^*\to H^{*-1}(\Omega X)$ be the cohomology suspension map. Then for $r>1$,*

$$\sigma^* R^{2^r+2^{r+1}k-1}\subseteq \mathrm{Sq}^{2^r}PH^*(\Omega X)$$

and for $r=1$

$$\sigma^* R^{4k+1}\subseteq \mathrm{Sq}^2 F_2$$

where F_2 is the vector space spanned by $PH^*(\Omega X)$, and two fold products of suspensions.

Theorem 3 [9]. (a) *There are no even indecomposables in H^*.*

(b) $\mathrm{Tor}_{H^*(X)}(\mathbb{Z}_2,\mathbb{Z}_2)\cong H^*(\Omega X)$ *as coalgebras and, therefore, for $f>1$*

$$PH^{2^f-2}(\Omega X)=\sigma^* QH^{2^f-1}(X).$$

(c) *The Bockstein sequence for H^* collapses at E_2.*

Recall that the Bockstein spectral sequence establishes a relation between $H^*(X;\mathbb{Z}_2)$ and $H^*(X;\mathbb{Q})$. The following result is due to Browder [3].

Theorem 4. $\dim_{\mathbb{Q}} QH^{\text{odd}}(X;\mathbb{Q}) = \dim_{\mathbb{Z}_2} QH^{\text{odd}}(X;\mathbb{Z}_2)$.

To analyze the relationship between the mod 2 cohomology and the rational cohomology, we will trace through the mod 2 Bockstein sequence and see how generators are lost and created.

First note that there are no even generators in the mod 2 or rational cohomology, by Theorem 3 and the rational Borel structure theorem. Therefore,

$$H^* = \bigotimes_i \wedge(x_i) \bigotimes_j \mathbb{Z}_2[y_j]/y_j^{2^{f_j}}$$

where the x_i and y_j have odd degree.

The biprimitive form of H^* is

$${}_0EE_0H^* = \bigotimes_{i=1}^{l} \wedge(x_i) \bigotimes_{j=1}^{m} \left[\wedge(y_j) \bigotimes_{n=1}^{f_j-1} \wedge(y_j^{2^n})\right].$$

It is well known that all even primitives are infinite cycles in the biprimitive sequence and since ${}_\infty EE_\infty H^*$ is exterior on odd degree generators, the even primitives must be boundaries at some stage of the spectral sequence. By [3], the net effect is the following:

To kill off $y_j^{2^n}$ where $\deg y_j = 2l_j+1$, we lose a generator in degree $2^n(2l_j+1)-1$ and gain a generator in degree $2^{n+1}(2l_j+1)-1$. Since n ranges between 1 and f_j-1, at ${}_\infty EE_\infty$ there is a new odd generator in degree $2^{f_j}(2l_j+1)-1$, and a generator in degree $2(2l_j+1)-1$ is lost.

We obtain the following formulae: If

$$\dim P^{4l+2}{}_0EE_0 + \dim P^{4l+1}{}_\infty EE_\infty = \dim P^{4l+1}{}_0EE_0.$$

Let q be the number of y_js of degree $2l+1$ that are truncated at height 2^f. Then

$$\dim P^{2^f(2l+1)-1}{}_\infty EE_\infty = \dim P^{2^f(2l+1)-1}{}_0EE_0 + q.$$

Finally note that ${}_\infty EE_\infty$ is the biprimitive form of $H^*(X;\mathbb{Q})$ and is isomorphic as algebras. From this and the above formulas, we obtain the following theorem.

Theorem 5. (a) *$QH^{2^f(2n+1)-1}(X;\mathbb{Q})$ is nontrivial for $f>1$ if and only if either there is a nontrivial truncated polynomial algebra of the form $\mathbb{Z}_2[y]/y^{2^f} \subseteq H^*$ where degree $y=2n+1$ or $QH^{2^f(2n+1)-1}(X;\mathbb{Z}_2)$ is nontrivial.*

(b) $\dim QH^{4n+1}(X;\mathbb{Q}) \leqslant \dim QH^{4n+1}(X;\mathbb{Z}_2)$.

(c) $\dim QH^{2^f-1}(X;\mathbb{Q}) = \dim QH^{2^f-1}(X;\mathbb{Z}_2)$.

(d) $\dim QH^{4n-1}(X;\mathbb{Q}) \geqslant \dim QH^{4n-1}(X;\mathbb{Z}_2)$.

Theorem 6. *If X is an H-space with the rational cohomology of the Lie group G_2 or F_4 then X has the* mod 2 *cohomology of G_2 or F_4, and $H^*(X;\mathbb{Z}_2)$ must be primitively generated.*

Proof. Consider first G_2. Suppose $H^*(X;\mathbb{Q})=H^*(G_2;\mathbb{Q})=\bigwedge(x_3,x_{11})$. By Theorem 5(a) either $Q^{11}\neq 0$ or $\mathbb{Z}_2[x_3]/x_3^4\subseteq H^*$. But by Theorem 2(c), if $Q^{11}\neq 0$ then

$$\sigma^* R^{11}\subseteq \mathrm{Sq}^4\, PH^6(\Omega X)=\mathrm{Sq}^4\sigma^* Q^7.$$

The last equality comes from Theorem 3(b). Since $\sigma^*: Q^{\mathrm{odd}}\to PH^{\mathrm{even}}(\Omega X)$ is monic we obtain $Q^{11}=\mathrm{Sq}^4\, Q^7$. But by Theorem 5(c) this implies $QH^7(X;\mathbb{Q})\neq 0$ which is a contradiction.

Therefore $\mathbb{Z}_2[x_3]/x_3^4\subseteq H^*$. But that implies $0\neq \mathrm{Sq}^2\, x_3=x_5\in R$. By Theorem 4, x_3, x_5 are all the generators, so $x_5^2=0=\mathrm{Sq}^1\,\mathrm{Sq}^4\, x_5$. The module R is now completely determined. So we have

$$H^*(X;\mathbb{Z}_2)=(\mathbb{Z}_2[x_3]/x_3^4)\otimes\bigwedge(x_5).$$

Now consider an H-space X with

$$H^*(X;\mathbb{Q})=H^*(F_4;\mathbb{Q})=\bigwedge(x_3,x_{11},x_{15},x_{23}).$$

By the same argument as before

$$\mathbb{Z}_2[x_3]/x_3^4\subseteq H^*.$$

Now if $x_5^2\neq 0$, by Theorem 5(a), $QH^{2f(5)-1}(X;\mathbb{Z})\neq 0$ for some $f>1$. This is impossible. Therefore $x_5^2=0$. Again by Theorem 5(a) $Q^{23}\neq 0$ since otherwise, we would also have $\mathbb{Z}_2[x_3]/(x_3)^8\subseteq H^*$. This would imply $\dim_{\mathbb{Z}_2} Q^3>\dim_{\mathbb{Q}} QH^3(X:\mathbb{Q})$ which contradicts Theorem 5(c). So we must have

$$H^*(X;\mathbb{Z}_2)=(\mathbb{Z}_2[x_3]/x_3^4)\otimes\bigwedge(x_5,x_{15},x_{23}).$$

Theorem 2 forces the Steenrod algebra structure to be

$$\mathrm{Sq}^2\, x_3=x_5 \quad \textit{for both } G_2 \textit{ and } F_4,$$

$$\mathrm{Sq}^8\, x_{15}=x_{23} \quad \text{for } F_4.$$

By Theorem 1, since R is a coalgebra, $\bar\Delta x_{15}\in\xi H^*\otimes R^{\mathrm{odd}}$.

Since R^3 and R^{15} generate H^* over the Steenrod algebra and these restrictions force R^3 and R^{15} to be primitive, $H^*(X;\mathbb{Z}_2)$ must be primitively generated in both cases. □

Theorem 7. *If an H-space X has the rational cohomology of the exceptional group E_i for $i=6$, 7 or 8, then X has the* mod 2 *cohomology of E_i, as algebras over the Steenrod algebra. Furthermore, fifteen dimensional generators cannot be primitive.*

Proof.

Case 1. E_6. Suppose $H^*(X;\mathbb{Q})=H^*(E_6;\mathbb{Q})=\bigwedge(x_3,x_9,x_{11},x_{15},x_{17},x_{23})$. Exactly as in the proof for F_4 in Theorem 6

$$H^*\supseteq(\mathbb{Z}_2[x_3]/x_3^4)\otimes\bigwedge(x_5)\otimes\bigwedge(x_{15},x_{23}).$$

By Theorem 5(a) and (b), $\bigwedge(x_9,x_{17})\subseteq H^*$. So R is completely determined and

$$H^*=(\mathbb{Z}_2[x_3]/x_3^4)\otimes\bigwedge(x_5,x_9,x_{15},x_{17},x_{23}).$$

Case 2. E_7. Suppose $H^*(X;\mathbb{Q}) = H^*(E_7;\mathbb{Q}) = \bigwedge(x_3, x_{11}, x_{15}, x_{19}, x_{23}, x_{27}, x_{35})$. Again

$$H^* \supseteq (\mathbb{Z}_2[x_3]/x_3^4) \quad \text{and} \quad Q^{11} = 0.$$

Now consider x_{19}. If $Q^{19} \neq 0$ then by Theorem 2 $Q^{19} = \mathrm{Sq}^8 Q^{11} = 0$. Therefore, $H^* \supseteq \mathbb{Z}_2[x_5]/x_5^4$ and $Q^{19} = 0$.

By a similar argument $Q^{35} = \mathrm{Sq}^{16}\, Q^{19} = 0$. Hence $\mathbb{Z}_2[x_9]/x_9^4 \subseteq H^*$. The argument for F_4 implies $\bigwedge(x_{15}, x_{23}) \subseteq H^*$. Finally $\bigwedge(x_{27}) \subseteq H^*$ since otherwise $Q^7 \neq 0$ which cannot hold by Theorem 5(c). So

$$H^* = \frac{\mathbb{Z}_2[x_3, x_5, x_9]}{x_3^4, x_5^4, x_9^4} \otimes \bigwedge(x_{15}, x_{17}, x_{23}, x_{27}).$$

Case 3. E_8. Suppose $H^*(X;\mathbb{Q}) = H^*(E_8;\mathbb{Q}) = \bigwedge(x_3, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}, x_{59})$. By Theorem 5, $Q^{31} = 0$ so $Q^{59} = 0$ and $Q^{47} = 0$. Similarly using Theorem 2(c) and 3(b), if $Q^{39} \neq 0$ then

$$\sigma^* Q^{39} = \mathrm{Sq}^8\, PH^{30}(\Omega X) = \mathrm{Sq}^8\, \sigma^* Q^{31} = 0,$$

$$\sigma^* Q^{35} = \mathrm{Sq}^4\, \sigma^* Q^{31} = 0.$$

But $\sigma^* : Q^{\mathrm{odd}} \to PH^{\mathrm{even}}(\Omega X)$ is monic. So $Q^{39} = 0$ and $Q^{35} = 0$. Theorem 5(a) implies

$$H^* \supseteq \mathbb{Z}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4).$$

Now $\mathrm{Sq}^8\, x_9 = x_{17} \neq 0$ since $x_9^2 \neq 0$, but $x_{17}^2 = 0$ since otherwise $QH^{2^f(17)-1}(X;\mathbb{Q}) \neq 0$ for $f > 1$. Similar arguments show $H^* \supseteq \bigwedge(x_{23}, x_{27}, x_{29})$. So we conclude

$$H^* = \frac{\mathbb{Z}_2[x_3, x_5, x_9, x_{15}]}{x_3^{16}, x_5^8, x_9^4, x_{15}^4} \otimes \bigwedge(x_{17}, x_{23}, x_{27}, x_{29}).$$

The action of the Steenrod algebra, by Theorem 2, must be

$$\mathrm{Sq}^2\, x_3 = x_5, \quad \mathrm{Sq}^4\, x_5 = x_9, \quad \mathrm{Sq}^8\, x_9 = x_{17}, \quad \mathrm{Sq}^8\, x_{15} = x_{23}$$

for E_6, E_7, E_8.

$$\mathrm{Sq}^4\, x_{23} = x_{27} \quad \text{for } E_7, E_8,$$

$$\mathrm{Sq}^2\, x_{27} = x_{29} \quad \text{for } E_8.$$

By Theorem 2(c) $\sigma^* x_{17} \in \mathrm{Sq}^2\, F_2$. Either

$$\sigma^* x_{17} \in \mathrm{Sq}^2 \sigma^* Q^{15}$$

or

$$\sigma^* x_{17} \in \mathrm{Sq}^2 \quad \text{(two-fold products of suspensions)}.$$

In the second case this would imply $\sigma^*(x_{17}) = y_8^2$ and $y_8 \in \mathrm{Sq}^2\, PH^6(\Omega X)$. But since $PH^6(\Omega X) = 0$ by Theorem 3(b) and 5(c), we must have

$$\sigma^* x_{17} = \mathrm{Sq}^2\, \sigma^* x_{15} \quad \text{or} \quad x_{17} = \mathrm{Sq}^2\, x_{15}$$

for the Lie groups E_6, E_7 and E_8.

Now if x_{15} is primitive, the action of the Steenrod algebra would force the cohomology of E_i to be primitively generated. By Thomas [11, 12] this would imply $x_9 = \mathrm{Sq}^2 x_7$ which is a contradiction. Therefore, $\bar{\Delta}x_{15}$ must have a nonzero summand. It is not too difficult to show that $\bar{\Delta}x_{15}$ must have at least $x_3^2 \otimes x_9$ as a nonzero summand. We refer the reader to [6, 11] for details. □

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